Non-relativistic quantum systems on topological defects space-times

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Abstract

We study the behavior of non-relativistic quantum particles interacting with different potentials in the space-times generated by a cosmic string and a global monopole. We find the energy spectra in the presence of these topological defects and show how they differ from their free-space-time values.

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I. Introduction

The study of quantum systems under the influence of a gravitational field has been an exciting research field. Along this line of research the hydrogen atom, for example, has been studied in particular curved space-times [1,2]. An atom placed in a gravitational field will be influenced by its interaction with the local curvature as well as with the topology of the space-time. As a result of this interaction, an observer at rest with respect to the atom would see a change in its spectrum. This shift in the energy of each atomic level would depend on the features of the space-time. The problem of finding these shifts [3] in the energy levels under the influence of a gravitational field is of considerable theoretical interest as well as possible observational. These shifts in the energy spectrum due to the gravitational field are different from the ones produced by

the electromagnetic field present, for example, near white dwarfs and neutron stars. In fact, it was already shown that in the Schwarzschild geometry, the level spacing of the gravitational effect is different from that of the well-known first order Stark and Zeeman effects, and therefore, in principle, it would be possible to separate the electromagnetic and gravitational perturbations of the spectrum [3]. Other investigation concerning this interesting subject is to use loosely bound Rydberg atoms in curved space-time to study the gravitational shift in the energy spectrum [4].

The first experiment which showed the gravitational effect on a wave function was performed by Colella, Overhauser and Werner [5] in which they measured the quantum mechanical phase difference of two neutron beams induced by a gravitational field. Another gravitational effect that appears in quantum interference is the neutrino oscillations [6] which has been discussed recently.

The general theory of relativity, as a metric theory, predict that gravitation is manifested as curvature of space-time. This curvature is characterized by the Riemann tensor $R^{\alpha}_{\beta\gamma\delta}$. It is of interest to know how the curvature of space-time at the position of the atom affects its spectrum. On the other hand, we know that there are connections between topological properties of the space and local physical laws in such a way that the local intrinsic geometry of the space is not sufficient to describe completely the physics of a given system. Therefore, it is also important to investigate the role played by a nontrivial topology, for example, on a quantum system. As examples of these investigations we can mention the calculation of the topological scattering amplitude in the context of quantum mechanics on a cone [7] and the interaction of a quantum system with conical singularities [8,9].

Therefore, the problem of finding how the energy spectrum of an atom placed in a gravitational field is perturbed by this background has to take into account the geometrical and topological features of the space-time under consideration and in this way we should emphasize that when a quantum system is embedded in a curved space-time it is influenced by its structure and topology. In other words, the dynamic of atomic systems is determined by the curvature at the position of the atom and also by

the topology of the background space-time.

According to standard quantum mechanics, the motion of a charged particle can be influenced by electromagnetic fields in regions from which the particle is rigorously excluded [10]. In this region the electromagnetic field vanishes. This phenomenon has come to be called Aharonov-Bohm effect after a seminal paper by Aharonov and Bohm [10]. It was shown that in the quantum scattering problem the background leads to a non-trivial scattering, which was confirmed experimentally [11].

The analogue of the electromagnetic Aharonov-Bohm effect set up is the background space-time of a cosmic string [12] in which the geometry is flat everywhere apart from a symmetry axis.

Cosmic strings [12] and monopoles [13] are exotic topological defects [14] which may have been formed at phase transitions in the very early history of the Universe. Up to the moment no direct observational evidence of their existence has been found, but the richness of the new ideas they brought along to general relativity seems to justify the interest in the study of these structures.

The gravitational field of a cosmic string is quite remarkable; a particle placed at rest around a straight, infinite, static string will not be attracted to it; there is no local gravity. The space-time around a cosmic string is locally flat but not globally. The external gravitational field due to a cosmic string may be approximately described by a commonly called conical geometry. Due to this conical geometry a cosmic string can induce several effects like, for example, gravitational leasing [15], pair production [16], electrostatic self-force [17] on an electric charge at rest, bremsstrahlung process [18] and the so-called gravitational Aharonov-Bohm effect [21].

The space-time of a point global monopole has also some unusual properties. It possesses a deficit solid angle $\Delta = 32\pi^2 G\eta^2$, η being the energy scale of symmetry breaking. Test particles in this space-time experiences a topological scattering by an angle $\pi\frac{\Delta}{2}$ irrespective of their velocity and their impact parameter. The effects produced by the point global monopole are due to the deficit solid angle which determines the curvature and the topological features of this space-time.

The aim of this paper is to study the energy shift associated with a non relativistic quantum particle interacting with different potentials in the space-times generated by a cosmic string and a global monopole.

In the cosmic string case, we consider the harmonic oscillator and the Coulomb potentials and determine how the nontrivial topology of this background space-time perturbs the energy spectrum. The influence of the conical geometry on the energy eigenvalues mani fests as a kind of gravitational Aharonov-Bohm effect for bound states [8,19], whose analogue in the electromagnetic case shows that [20] the bound state energy depend on the external magnetic flux in a region from which the electron is excluded.

In the case of a point global monopole we are concerned with a similar proposal. We will investigate the interactions of a non-relativistic quantum particle with the Kratzer [22] and the Morse [23] molecular potentials in this background space-time. In this case we also determine the shifts in the energy levels.

These modifications in the energy spectra as compared to the simplest situation of empty flat Minkowski space-time, could be used, in principle, as a probe of the presence of these defects in the cosmos. The possibility of using this effect for such a purpose is interesting because at present our observational knowledge of the existence of such defects is quite limited and this effect offers one more possibility to detect these topological defects. In fact to produce observable modifications in the spectra from the astrophysical point of view, one needs an huge number of particles in the states we are studying, otherwise the contribution to real spectral effects is thus not likely to be sufficiently strong to be observed.

In order to do these studies let us consider that a non-relativistic particle living in a curved space-time is described by the Schrödinger equation which should take the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mu}\nabla^2_{LB}\psi + V\psi, \tag{1}$$

where ∇^2_{LB} is the Laplace-Beltrami operator the covariant version of the Laplacian

given by $\nabla^2_{LB} = g^{-\frac{1}{2}}\partial_i\left(g^{ij}g^{\frac{1}{2}}\partial_j\right)$, with $i, j = 1, 2, 3$; $g = \det\left(g_{ij}\right)$; μ is the mass of the particle and V is an external potential. Throughout this paper we will use units in which $c = 1$.

II. Coulomb potential in the space-time of a cosmic string

In what follows we analyze the energy level of a particle in the presence of a Coulomb potential in the space-time of a cosmic string. To do this let us consider the exterior metric of an infinitely long straight and static string in spherical coordinates. It reads as

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + \alpha^2 r^2 \sin^2\theta d\varphi^2, \tag{2}$$

with $0 < r < \infty$, $0 < \theta < \pi$, $0 \leq \varphi \leq 2\pi$. The parameter $\alpha = 1 - 4G\bar{\mu}$ runs in the interval $[0, 1]$, $\bar{\mu}$ being the linear mass density of the cosmic string. In the special case $\alpha = 1$ we obtain the Minkowski space in spherical coordinates.

The time-independent Schrödinger equation in this case is

$$\begin{aligned}&-\frac{\hbar^2}{2\mu r^2}\left[2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\alpha^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\psi\left(r,\theta,\varphi\right)\\ &V(r)\psi\left(r,\theta,\varphi\right) = E\psi\left(r,\theta,\varphi\right).\end{aligned} \tag{3}$$

Let us assume that the eigenfunctions have the form

$$\psi(r,\theta,\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}R(r)\Theta(\theta) \tag{4}$$

then, we get the following equations for the radial and angular parts

$$-\frac{\hbar^2}{2\mu}\frac{d^2u(r)}{dr^2} + V(r)_{eff}u(r) = Eu(r), \tag{5}$$

where we changed the function $R(r)$ by $u(r)$ introducing $u(r) = rR(r)$ and $V_{eff}(r)$ is the effective potential experienced by the particle and given by

$$V_{eff}(r) = V(r) - \frac{\lambda}{r^2}. \tag{6}$$

and

$$\left(1-\xi^2\right)\frac{d^2F(\xi)}{d\xi^2}-2\xi\frac{dF(\xi)}{d\xi}-\left(\frac{2\mu\lambda}{\hbar^2}+\frac{m^2}{\alpha^2\left(1-\xi^2\right)}\right)F(\xi)=0, \tag{7}$$

where we introduced the change of variables $\xi = \cos\theta$ and $F(\xi) \equiv \Theta(\theta)$. Equation (7) is the generalized associated Legendre equation. It becomes an equation with eigenvalue $\frac{2\mu\lambda}{\hbar^2}$, if we demand that the solution be finite at the singular points $\xi = \pm 1$. Its examination follows a conventional pattern.

A convenient method to obtain the solutions of Eq. (7) is by the investigation of its behavior at the singular points $\xi = \pm 1$. Doing this, the accepted solution may have the form

$$F(\xi)=\left(1-\xi^2\right)^{\frac{m}{2\alpha}}G(\xi), \tag{8}$$

where G is analytic in all space except when $\xi \to \pm\infty$, and is different from zero for $\xi = \pm 1$.

From Eqs. (7) and (8), we get

$$\left(1-\xi^2\right)\frac{d^2G(\xi)}{d\xi^2}-2(m_{(\alpha)}+1)\xi\frac{dG(\xi)}{d\xi}-\left(m_{(\alpha)}^2+m_{(\alpha)}+\bar{\lambda}\right)G(\xi)=0, \tag{9}$$

where

$$m_{(\alpha)}=\frac{m}{\alpha};\ \bar{\lambda}=\frac{2\mu\lambda}{\hbar^2}.$$

Expanding the regular solution of Eq. (9) in a power series we get the recursion relation

$$a_{n+2}=\frac{n\left(n-1\right)+2\left(\bar{m}+1\right)n+\bar{\lambda}+\bar{m}\left(\bar{m}+1\right)}{\left(n+1\right)\left(n+2\right)}a_n, \tag{10}$$

where n is an integer ≥ 0, and $G(\xi)$ diverges at $|\xi| = 1$. In order to have acceptable eigenfunctions, the series must be terminate at some finite value of n. According to Eq. (10), this will happen if $\bar{\lambda}$ has the value

$$\Rightarrow \bar{\lambda}=-l_{(\alpha)}\left(l_{(\alpha)}+1\right), \tag{11}$$

where

$$l_{(\alpha)}=m_{(\alpha)}+n.$$

Therefore, Eq. (7) turns into

$$\left(1-\xi^2\right)\frac{d^2F(\xi)}{d\xi^2}-2\xi\frac{dF(\xi)}{d\xi}+l_{(\alpha)}(l_{(\alpha)}+1)F(\xi)-\frac{m^2}{\alpha^2(1-\xi^2)}F(\xi)=0, \tag{12}$$

which corresponds to a generalized Legendre equation in the sense that now $l_{(\alpha)}$ and $m_{(\alpha)}$ are not necessarily integers. Its solutions are thus given by

$$F_{l_{(\alpha)}}^{m_{(\alpha)}}(\xi)(\xi)=P_{l_{(\alpha)}}^{m_{(\alpha)}}(\xi)=\frac{1}{2^{l_{(\alpha)}}l_{(\alpha)}!}\left(1-\xi^2\right)^{\frac{m_{(\alpha)}}{2}}\frac{d^{m_{(\alpha)}+l_{(\alpha)}}}{d\xi^{m_{(\alpha)}+l_{(\alpha)}}}\left[(\xi^2-1)^{l_{(\alpha)}}\right]. \tag{13}$$

Now, substituting Eq. (11) into (5) and considering $V(r)=-\frac{k}{r}$, we find

$$\frac{d^2u(r)}{dr^2}+2k\frac{\mu}{r\hbar^2}u(r)-\bar{\beta}^2u(r)-\frac{1}{r^2}\left[l_{(\alpha)}(l_{(\alpha)}+1)\right]u(r)=0, \tag{14}$$

where

$$\bar{\beta}^2=-\frac{2\mu E_{n_r}}{\hbar^2};\ E_{n_r}<0. \tag{15}$$

Equation (14) is a confluent hypergeometric equation whose solution is given by

$$u(r)={}_1F_1\left(l_{(\alpha)}+1-\frac{k^2\mu}{\bar{\beta}\hbar^2},2+2l_{(\alpha)};2\bar{\beta}r\right). \tag{16}$$

This function is divergent, unless

$$1+l_{(\alpha)}-\frac{\mu k^2}{\bar{\beta}\hbar^2}=-n_r;\ n_r=0,1,2...\ . \tag{17}$$

Then, from the previous condition we find the energy eigenvalues

$$E_{n_r}=-\frac{\mu k^2}{2\hbar^2}\left[l_{(\alpha)}+n_r'\right]^{-2};\ n_r'=1,2,3..., \tag{18}$$

where $n_r'=1+n_r$. Note that the energy levels become more and more spaced as α tends to 1, which means that the shift in the energy levels due to the presence of the cosmic string increases with the increasing of the angular deficit.

From the expression for the energy given by Eq. (18) we can notice that the levels without a z-component of the angular momentum are not shifted relative to the Minkowski case. Except these levels all the other ones are degenerated.

As an estimation of the effect of the cosmic string on the energy shift, let us consider $\alpha\cong 0.999999$, which corresponds to GUT cosmic strings. In this case the energy of the

particle in the presence of the cosmic string decreases of about $4\times10^{-3}\%$ as compared with the flat space-time value.

III. Harmonic oscillator in the space-time of a cosmic string

The line element corresponding to the cosmic string space-time is given in cylindrical coordinates by [24]

$$ds^2 = -dt^2 + d\rho^2 + \alpha^2\rho^2 d\theta^2 + dz^2, \tag{19}$$

where $\rho \geq 0$ and $0 \leq \theta \leq 2\pi$. The string is situated on the z-axis. In the special case $\alpha = 1$ we obtain the Minkowski space in cylindrical coordinates. This metric has a cone-like singularity at $\rho = 0$. In other words, the curvature tensor of the metric (19), considered as a distribution, is of the form

$$R^{12}_{\ \ 12} = 2\pi\frac{\alpha-1}{\alpha}\delta^2(\rho), \tag{20}$$

where $\delta^2(\rho)$ is the two-dimensional Dirac δ-function.

Now, let us consider the Schrödinger equation in the metric (19) which is given by

$$-\frac{\hbar^2}{2\mu}\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\alpha^2\rho^2}\partial_\theta^2 + \partial_z^2\right]\psi(t,\rho,\theta,z) + V(\rho,z)\psi(t,\rho,\theta,z) = i\hbar\frac{\partial}{\partial t}\psi(t,\rho,\theta,z), \tag{21}$$

where $V(\rho,z)$ is the interaction potential corresponding to a three-dimensional harmonic oscillator which is assumed to be

$$V(\rho,z) = \frac{1}{2}\mu w^2\left(\rho^2 + z^2\right). \tag{22}$$

We will now determine the eigenfunctions of the Schrödinger equation (Eq. (21)) with the interaction potential given by Eq. (22), by searching for solutions of the form

$$\psi(t,\rho,\theta,z) = \frac{1}{\sqrt{2\pi}}e^{-i\frac{Et}{\hbar}+im\theta}R(\rho)Z(z). \tag{23}$$

Equation (21) leads to two ordinary differential equations for $R(\rho)$ and $Z(z)$ which are given by

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{R(\rho)}\frac{d^2R(\rho)}{d\rho^2} + \frac{1}{R(\rho)\rho}\frac{dR(\rho)}{d\rho} - \frac{m^2}{\alpha^2\rho^2}\right] + \frac{1}{2}\mu w^2\rho^2 = \Omega \tag{24}$$

and

$$-\frac{\hbar^2}{2\mu Z(z)}\frac{d^2Z(z)}{dz^2}+\frac{1}{2}\mu w^2z^2=\varepsilon_z, \tag{25}$$

where Ω is a separation constant and such that

$$\Omega+\varepsilon_z=E. \tag{26}$$

Equation (25) is the Schrödinger equation for a particle in the presence of one-dimensional harmonic oscillator potential, and then we have the well-known results

$$\varepsilon_z=\left(n_z+\frac{1}{2}\right)\hbar w;\ n_z=0,1,2,..., \tag{27}$$

with

$$Z(z)=2^{-\frac{n_z}{2}}\left(n_z!\right)^{-\frac{1}{2}}\left(\frac{\mu w}{\hbar\pi}\right)^{\frac{1}{4}}e^{-\frac{\mu w}{2\hbar}z^2}H_{n_z}\left(\sqrt{\frac{\mu w}{\hbar}}z\right), \tag{28}$$

where H_{n_z} is the Hermite Polynomial.

Now, let us look for solutions of Eq. (24). Its solution can be written as

$$R(\rho)=\exp\left(-\frac{\tau}{2}\rho^2\right)\rho^{\frac{|m|}{\alpha}}F(\rho), \tag{29}$$

where $\tau=\frac{mw}{\hbar}$ and

$$F(\rho)={}_1F_1\left(\frac{1}{2}+\frac{|m|}{2\alpha}-\frac{\mu\Omega}{2\hbar^2\tau},\frac{A}{2};\tau\rho^2\right) \tag{30}$$

is the degenerate hypergeometric function, with $A=1+2\frac{|m|}{\alpha}$.

In order to have normalizable wave-function, the series in Eq. (30) must be a polynomial of degree n_ρ, and therefore

$$\frac{1}{2}+\frac{|m|}{2\alpha}-\frac{\mu\Omega}{2\hbar^2\tau}=-n_\rho;\ n_\rho=0,1,2,.... \tag{31}$$

With this condition, we obtain the following energy eigenvalues

$$\Omega=\hbar w\left(1+\frac{|m|}{\alpha}+2n_\rho\right). \tag{32}$$

If we substitute Eqs. (32) and (27) into (26) we get, finally, the energy eigenvalues

$$E_N=\hbar w\left(N+\frac{|m|}{\alpha}+\frac{3}{2}\right), \tag{33}$$

where $N = 2n_\rho + n_z$.

Therefore, the complete eigenfunctions are then given by

$$\psi(t,\rho,\theta,z) = C_{Nm} e^{-iE_N\frac{t}{\hbar}} e^{-\frac{\tau}{2}\rho^2} \rho^{\frac{|m|}{\alpha}} F_1\left(\frac{1}{2} + \frac{|m|}{2\alpha} - \frac{\mu\Omega}{2\hbar^2\tau}, \frac{A}{2}; \tau\rho^2\right)$$
$$e^{im\theta} 2^{-\frac{n_z}{2}} (n_z!)^{-\frac{1}{2}} \left(\frac{\mu w}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{\mu w}{2\hbar}z^2} H_{n_z}\left(\sqrt{\frac{\mu w}{\hbar}} z\right), \tag{34}$$

where C_{Nm} is a normalization constant. It is worth calling attention to the fact that the presence of the cosmic string breaks the degeneracy of the energy levels.

Note that in the limit $\alpha \to 1$ we get the results corresponding to the three-dimensional harmonic oscillator in flat space-time as it should be.

In the case under consideration the shift in the energy spectrum between the first two levels in this background increases of about $10^{-5}\%$ as compared with the flat Minkowski space-time case.

Now, if we consider the spherical harmonic oscillator potential $V(r) = \frac{1}{2}\mu\omega^2 r^2$, we can do similar calculations using the metric of the cosmic string in spherical coordinates, in which case we obtain the following expression for the energy spectra

$$E_{n,l(\alpha)} = \hbar w \left(l_{(\alpha)} + 2n + \frac{3}{2}\right);\ n = 0, 1, 2, ..., \tag{35}$$

which is a result similar to the previous one given by Eq. (33).

IV. Kratzer potential in the space-time of a global monopole

The solution corresponding to a global monopole in a $O(3)$ broken symmetry model has been investigated by Barriola and Vilenkin [13].

Far away from the global monopole core we can neglect the mass term and as a consequence the main effects are produced by the solid deficit angle. The respective metric in the Einstein theory of gravity can be written as [13]

$$ds^2 = -dt^2 + dr^2 + b^2 r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{36}$$

where $b^2 = 1 - 8\pi G\eta^2$, η being the energy scale of symmetry breaking and r, θ and φ are usual spherical coordinates.

Now, let us consider a particle interacting with a Kratzer potential given by

$$V(r) = -2D\left(\frac{A}{r} - \frac{1}{2}\frac{A^2}{r^2}\right), \tag{37}$$

where A and D are positive constants, and placed in the background space-time given by metric (36).

In order to determine the energy spectrum let us write the Schrödinger equation in the background space-time of a global monopole. Then, we get

$$-\frac{\hbar^2}{2\mu b^2 r^2}\left[2rb^2\frac{\partial}{\partial r} + b^2 r^2\frac{\partial^2}{\partial r^2} - \mathbf{L}^2 - 2D\left(\frac{A}{r} - \frac{1}{2}\frac{A^2}{r^2}\right)\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{38}$$

where $\mathbf{L}$ is the usual orbital angular momentum operator. We begin by using the standard procedure for solving Eq. (38) and assume that the eigenfunctions can be written as

$$\psi_{m,l}(\mathbf{r}) = R_l(r)Y_l^m(\theta,\varphi). \tag{39}$$

Substitution of Eq. (39) into Eq. (38) leads to

$$-\frac{\hbar^2}{2\mu}\frac{d^2 g_l(r)}{dr^2} - 2D\left(\frac{A}{r} - \frac{1}{2}\frac{A^2}{r^2}\right)g_l(r) + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{b^2 r^2}g_l(r) = Eg_l(r), \tag{40}$$

where $g_l(r) = rR_l(r)$.

The solution of Eq. (40) can be written as

$$g_l(r) = r^{\lambda_l}e^{-\bar{\beta}r}F_l(r) \tag{41}$$

where

$$\lambda_l = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\left(\frac{2\mu DA^2}{\hbar^2} + \frac{l\,(l+1)}{b^2}\right)}. \tag{42}$$

Substituting Eq. (41) into Eq. (40) and making use of Eqs. (15) and (42) we obtain the equation for $F(z)$

$$z\frac{d^2F(z)}{dz^2} + (2\lambda_l - z)\frac{dF(z)}{dz} - \left(\lambda_l - \frac{2mAD}{\bar{\beta}\hbar^2}\right)F(z) = 0, \tag{43}$$

where $z = 2\bar{\beta}r$.

The solution of this equation is the confluent hypergeometric function ${}_1F_1\left(\lambda_l - \frac{\gamma^2}{\bar{\beta}A}, 2\lambda_l \ ; \ 2\bar{\beta}r\right)$, with $\gamma^2 = \frac{2\mu D A^2}{\hbar^2}$.

Therefore, the solution for the radial function $g_l(r)$ is given by

$$g_l(r) = r^{\lambda_l} e^{-\bar{\beta}r} {}_1F_1\left(\lambda_l - \frac{\gamma^2}{\bar{\beta}A}, 2\lambda_l \ ; \ 2\bar{\beta}r\right). \tag{44}$$

In order to make $g_l(r)$ vanishes for $r \to \infty$, the confluent hypergeometric function may increase not faster than some power of r, that is, the function must be a polynomial. Hence

$$\lambda_l - \frac{\gamma^2}{\bar{\beta}A} = -\bar{n}_r, \ \bar{n}_r = 0, 1, 2, \ldots \ . \tag{45}$$

With this condition we find that the eigenvalues are given by

$$E_{l,\bar{n}_r} = -\frac{\hbar^2}{2\mu A^2}\gamma^4 \left(\bar{n}_r + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{l\,(l+1)}{b^2} + \gamma^2}\right)^{-2} \tag{46}$$

From this expression, we see that when $b = 1$ we recover the well-known result corresponding to a particle submitted to the Kratzer potential [22] as it should be.

It is worth noticing from expression for the energy given by Eq. (46) that even in the case in which the $z-$component of the angular momentum vanishes the energy level is shifted relative to the Minkowski case.

As an estimation of the effect of the global monopole on the energy spectrum, let us consider a stable global monopole configuration for which $\eta = 0.19 m_p$, where m_p is the Planck mass. In this situation the shift in the energy spectrum between the first two levels in this space-time decreases of about 82% as compared with the Minkowski space-time. For symmetry breaking at grand unification scale, the typical value of $8\pi G\eta^2$ is around 10^{-6} and in this case the energy shift decreases of about 1%.

V. Morse potential in the space-time of a global monopole

Now, let us consider the Morse potential [23] which is used to describe the vibrations of a diatomic molecule and reads as

$$V(r) = D\left(e^{2\beta x} - 2e^{-\beta x}\right); \ \ x = \frac{r - r_0}{r_0}; \ \beta > 0. \tag{47}$$

In order to calculate and analyze the modifications in the energy spectrum by the presence of a global monopole, let us consider a particular situation and expand this potential around $r = r_0$ (or $x = 0$) which leads to

$$V(r') \cong -D + \frac{1}{2}Mw^2 r'^2, \tag{48}$$

where $r - r = r'$ and the frequency ω is defined by $\frac{1}{2}Mw^2 = \frac{D\beta^2}{r_0^2}$. This potential corresponds to the spherical harmonic oscillator potential minus a constant.

We now write the Schrödinger radial equation which is given by

$$\frac{1}{r'}\frac{d}{dr'}\left(r'^2\frac{dR}{dr'}\right) - \frac{M^2w^2r'^2}{\hbar^2}R(r') - l\frac{(l+1)}{b^2r'^2}R(r') + \frac{2M}{\hbar^2}E_M R(r') = 0. \tag{49}$$

Therefore, if we put

$$R(r') = \exp\left(-\frac{1}{2}M\frac{w}{\hbar}r'^2\right) r'^{-\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4}{b^2}l(l+1)}} F(r'), \tag{50}$$

we find

$$r'\frac{d^2F(r')}{dr'^2} + \left[c - \frac{2Mw}{\hbar}r'^2\right]\frac{dF(r')}{dr'} + \left[P - \frac{2Mw}{\hbar}\right]r'F(r') = 0, \tag{51}$$

where the parameters c and P are given by

$$c = 1 + \sqrt{1 + \frac{4}{b^2}l\,(l+1)};\; P = \frac{2ME_M}{\hbar^2} - \frac{Mw}{\hbar}\sqrt{1 + \frac{4}{b^2}l\,(l+1)} + \frac{2MD}{\hbar^2}.$$

By a change of variables,

$$Mwr'^2 = \rho, \tag{52}$$

Eq. (51) is transformed into

$$\rho\frac{d^2F(\rho)}{d\rho^2} + \frac{1}{2}\left[c - 2\frac{\rho}{\hbar}\right]\frac{dF(\rho)}{d\rho} + \left[\frac{P}{4Mw} - \frac{1}{2\hbar}\right]F(\rho) = 0. \tag{53}$$

Now, defining $\frac{\rho}{\hbar} = x$, we get

$$x\frac{d^2F(x)}{dx^2} + \left[\frac{c}{2} - x\right]\frac{dF(x)}{dx} + \left[\frac{P\hbar}{4Mw} - \frac{1}{2}\right]F(x) = 0, \tag{54}$$

whose solution is

$$F(x) = {}_1F_1\left(\frac{1}{2} - \frac{\hbar P}{4Mw}, \frac{c}{2}; x\right), \tag{55}$$

or, in terms of variable r', we have

$$F(r') = {}_1F_1\left(\frac{1}{2} - \frac{E_M}{2\hbar w} - \frac{D}{2\hbar w} + \frac{1}{4}\sqrt{1 + \frac{4}{b^2}l(l+1)}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{4}{b^2}l(l+1)}; \frac{Mwr'^2}{\hbar}\right). \tag{56}$$

Note that this solution is divergent, unless

$$\frac{1}{2} - \frac{E_M}{2\hbar w} - \frac{D}{2\hbar w} + \frac{1}{4}\sqrt{1 + \frac{4}{b^2}l(l+1)} = -n_M; \quad n_M = 0, 1, 2, 3, \ldots. \tag{57}$$

From Eq. (57) we find the energy levels which are given by

$$E_M = \hbar w\left[N_{lM} + \frac{3}{2}\right] - D, \tag{58}$$

where

$$N_{lM} = \frac{1}{2}\left(\sqrt{1 + \frac{4}{b^2}l\,(l+1)} - 1\right) + 2n_M. \tag{59}$$

An estimation of the modifications in the energy spectrum in this case show us a decrease in the energy eingenvalues of $3 \times 10^{-5}\%$ and 32% in the cases of GUT and stable monopoles, respectively, as compared to the flat Minkowski space-time case.

VI. FINAL REMARKS

In the space-time of a cosmic string we studied the behavior of a particle interacting with an harmonic oscillator and a Coulomb potential. The quantum dynamics of such a single particle depends on the nontrivial topological features of the cosmic string space-time. The presence of the defect shift the energy levels in both cases as compared to the flat Minkowski space-time one. It is interesting to observe that these shifts depend on the parameter that defines the angle deficit and therefore arise from the global features of the space-time of a cosmic string.

In the case of the Kratzer and Morse potentials in the space-time of a global monopole the shifts in the energy levels are due to the combined effects of the curvature and the nontrivial topology determined by the solid deficit angle corresponding to the space-time of a global monopole.

The magnitude of the modifications in the spectra of the quantum mechanical potentials by the presence of gravitational strings and monopoles are all measurable, in principle. In fact, to produce observable modifications at the astrophysical scale, one needs an huge number of particles in the states that we have studied. We can also have the possibility to get a very strong contribution to real spectral effects if the topological defects (cosmic strings and monopoles) occur in huge numbers, like in superfluid vortices in neutron stars which are expected to occur with densities of 10^6 vortices per square centimeter of stellar crossection.

Our estimation takes into account just a simple configuration with one cosmic string or one global monopole, but even in this simple situation we believe that astrophysical observations concerning the detectability of cosmic strings and monopoles in the Universe can be made although it seems difficult to provide these verifications with the current observational detectability.

An important application of these results could be found in the astrophysical context in which these modifications enters in the interpretation of the spectroscopical data and these could be used as a probe of the presence of a cosmic string or a global monopole in the Universe.

Finally, it is worth commenting that the study of a quantum system in a nontrivial gravitational background like the ones considered in this paper may shed some light on the problems of combining quantum mechanics and general relativity.

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